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Fluctuations of the Fermi condensate in ideal gases

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Abstract

We calculate numerically and analytically the grandcanonical fluctuation of the number of particles both, in the fermionic condensate and above it, for ideal Fermi systems of constant density of states. We compare the canonical fluctuations, obtained from the equivalent Bose condensate fluctuation, with the grandcanonical fermionic calculation. The fluctuations of the condensate are almost the same in the two ensembles with a small correction coming from the total particle number fluctuation in the grandcanonical ensemble. On the other hand, well below the condensation temperature, the number of particles above the condensate and its fluctuation are insensitive to the choice of ensemble.

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1. Introduction

Starting quite a long time ago, Auluck and Kothari [1], then May [2] and finally Viefers, Ravndal and Haugset [3] independently discovered that the specific heat of nonrelativistic ideal gases in two-dimensional (2D) boxes is unaffected by the exclusion statistics. This interesting result eventually did not receive the attention it deserved until 1995, when Lee [4–6] re-derived it by introducing an unified way of writing the thermodynamic properties of ideal gases in terms of polylogarithmic functions [7]. This formulation also represented an important extension of the Auluck and Kothari result and triggered further investigations (see, e.g., [8–10]). A Bose and a Fermi gas that have the same heat capacity are identical at the thermodynamic level, under canonical conditions. For this reason they are called *thermodynamically equivalent* [5]. If we denote by $C_V(T, V, N)$ the heat capacity of a system at temperature T , volume V and particle number N , then the heat capacities $C_{V,1}$ and $C_{V,2}$ of two thermodynamically equivalent systems are identical functions of T , V and N . Using this property, all the thermodynamic systems may be divided into equivalence classes [10].

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One can show that all the systems of ideal particles of the same constant density of single-particle states (DOS), obeying Bose, Fermi or even *fractional exclusion statistics* [11], belong to the same equivalence class [10, 12].

The equivalence between the Bose and the Fermi gases was critically examined by Pathria [9]. He showed that the Lee's unified formulation of 2D ideal gases does not hold anymore below the Bose–Einstein condensation temperature of the Bose gas. Apparently, the 2D (or, more exactly, constant DOS) thermodynamic equivalence holds only *above* the Bose–Einstein condensation temperature.

On the other hand, Crescimanno and Landsberg [13], and one of us [10] showed that there is a one-to-one mapping between microscopic configurations of bosons, fermions or particles obeying fractional exclusion statistics, in systems with the same, constant DOS, which preserves the total excitation energy, i.e. the energy of a given configuration minus the energy of the system in the lowest energy configuration. Based on this theorem, the thermodynamic equivalence of systems with equally spaced spectra should hold *in any detail at any temperature*. Thus Pathria's conclusion must be wrong. But what was overlooked there?

The method of mapping microscopic configurations between systems of different exclusion statistics, introduced in [10] for systems with constant DOS, was then extended for systems with any DOS [14]. Systems that map (or transform) into each other by this method are thermodynamically equivalent by construction. However, if we take a Fermi system, transform it into a Bose one, and then calculate independently the thermodynamic properties of these two systems by maximizing the entropy in each of them at constant internal energy, U , and particle number, N , then the thermodynamic equivalence that we started with is generally lost [14]. One has to keep in mind that the maximization of the entropy at constant U and N leads to the grandcanonical distribution on the energy levels. Thus, since the two gases are canonically equivalent, the obvious conclusion is that one or both of these Fermi and Bose grandcanonical distributions lead to results in disaccord with the canonical ensemble. The question which of these grandcanonical distributions is closer to the canonical distribution is very difficult to answer, since *ab initio* canonical calculations are hardly performable for general large systems.

1.1. The Fermi condensate

The Fermi condensate was introduced in [14, 15] for a system having N_0 lowest energy levels occupied, the $(N_0 + 1)$ th level empty and higher levels with arbitrary occupation numbers. Let us enumerate the single-particle energy levels, ϵ_i , as $\epsilon_0 \leq \epsilon_1 \leq \dots$. Then the probability, w_{N_0} , for such system to form a Fermi condensate reads as [15]:

$$w_{N_0} = \mathcal{Z}_{N_0} / \mathcal{Z}, \quad \mathcal{Z} = \sum_{N_0} \mathcal{Z}_{N_0},$$

$$\mathcal{Z}_{N_0} = \mathcal{Z}_{\text{ex}} \exp\left(-\beta \sum_{i=0}^{N_0-1} \epsilon_i + \beta \mu N_0\right), \quad \mathcal{Z}_{\text{ex}}(N_0, \beta, \beta \mu) = \prod_{i=N_0+1}^{\infty} (1 + \exp(-\beta(\epsilon_i - \mu))).$$
(1)

Here, \mathcal{Z} is the partition function of the system and $\mathcal{Z}_{\text{ex}}(N_0, \beta, \beta \mu)$ is the partition function of the levels $N_0 + 1, N_0 + 2, \dots$. The probability distribution (1) may have a maximum at some $N_0 = N_{\text{max}}$. The statistical interpretation of such a maximum is that in a physical system in contact with a particle reservoir, the lowest N_0 ($\approx N_{\text{max}}$) energy levels are always occupied. These N_0 particles form the *Fermi condensate*. At any finite temperature, N_0 is subject to fluctuations.

Configurations of fermions may be transformed as in [10, 14] into configurations of bosons. By this transformation, the N_0 degenerate fermions will be mapped onto the Bose–Einstein condensate (BEC) of the corresponding Bose system, and this is the reason for the name ‘Fermi condensate’. If the system is large enough, then the number of single-particle states in the interval $(\epsilon, \epsilon + \delta\epsilon)$ is approximated by $\sigma(\epsilon) \cdot \delta\epsilon$, where $\sigma(\epsilon)$ is the energy-dependent DOS. For canonical Bose systems of constant σ , the probability distribution of having N_0 particles in the condensate, $w_{N_0}^c$, has been studied in detail earlier (see [17] for a review). Then, by construction, $w_{N_0}^c$ also represents the canonical probability of having N_0 particles in the Fermi condensate. Moreover, at constant σ the maximum of the distribution w_{N_0} , N_{\max} , is close to the average canonical occupation of the ground state [16]. However, the two numbers are not the same because the distribution w_{N_0} is asymmetric.

The existence of the Fermi condensate restores also the Bose–Fermi equivalence challenged in [9]. According to section 5 of [14], for constant σ the grandcanonical Bose and Fermi distributions map onto each other. Therefore, the complete equivalence between the canonical Bose and Fermi gases is recovered if we separate the condensate (correspondent to BEC) from the Fermi gas and assume Fermi occupation numbers for the excited states [16]. For simplicity, we will refer to the fermions above the condensate as particles in the *thermally active layer*.

Below the condensation temperature, the Bose canonical and the grandcanonical ensembles are not equivalent because of the huge particle number fluctuations on the ground state. On the other hand, in Fermi systems particle number fluctuation is always microscopic, so as expected, the canonical and the grandcanonical ensembles are equivalent at any temperature. Nevertheless, the Fermi gas may be transformed into a Bose gas which below the condensation temperature possesses a condensate. Thus, one can ask whether the grandcanonical description (1) of the Fermi condensate and the bosonic canonical description are equivalent. Indeed, this is a new type of equivalence, first mentioned in [14], which seems not to hold in general.

Here, we shall discuss only the systems with constant σ and compare the grandcanonical and the canonical probability distributions, w_{N_0} and $w_{N_0}^c$. An important parameter of the system is the product $\sigma k_B T$. For large $\sigma k_B T$ we can do some analytical calculations, assuming that w_{N_0} has a Gaussian shape. As we shall see in section 2, this approximation is not good enough to evaluate the mean-square fluctuation of N_0 . To correct this and to extend our results to lower values of $\sigma k_B T$, in section 3, we calculate w_{N_0} , $w_{N_0}^c$ and the fluctuations numerically. We confirm that when $\sigma k_B T \rightarrow \infty$, the canonical mean-square fluctuation, $\langle \delta^2 N_0 \rangle^c$, tends to $\zeta(2)(\sigma k_B T)^2$, where $\zeta(x)$ is the Riemann zeta function. We also show that in the same limiting case the difference between the grandcanonical and the canonical fluctuations, $\langle \delta^2 N_0 \rangle - \langle \delta^2 N_0 \rangle^c$, tends to a constant value, 0.39. Thus, the mean-square fluctuation of N_0 is of the same order as the number of particles in the thermally active layer or the number of particles in the excited states in the Bose gas. This asymptotic behaviour is proved analytically at the end of section 3.

2. Analytical evaluation of the fluctuations

First, we analyse the distribution w_{N_0} analytically in the limit $\sigma k_B T \gg 1$. Transforming summations into integrals in the expression for $\log w_{N_0}$ following from equation (1), we obtain [15]

$$\log \mathcal{Z}_{N_0} = \left[-\beta \left(\sigma \frac{\epsilon_0^2}{2} - \epsilon_0 \right) + \beta \mu (\sigma \epsilon_0 - 1) \right] + \sigma \int_{\epsilon_0}^{\infty} d\epsilon \log[1 + \exp(-\beta(\epsilon - \mu))], \quad (2)$$

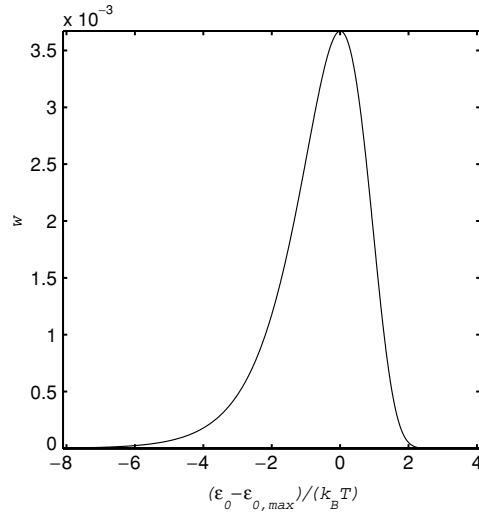


Figure 1. Probability distribution of w_{N_0} as a function of N_0 . The maximum of the probability distribution is located at N_{\max} , which is given by equation $(\mu - N_{\max}/\sigma)/k_B T = \log(\sigma k_B T)$. For this particular plot $\sigma k_B T = 100$.

where ϵ_0 is the energy of the N_0 th single-particle level, while the integral represents $\log \mathcal{Z}_{\text{ex}}(N_0, \beta, \beta\mu)$. The calculated probability distribution is shown in figure 1.

Since $\sigma \epsilon_0 = N_0$ and $\partial \log \mathcal{Z}_{N_0} / \partial N_0 = \sigma^{-1} (\partial \log \mathcal{Z}_{N_0} / \partial \epsilon_0)$, the value of ϵ_0 corresponding to the maximum of probability, ϵ_{\max} , is given by the equation

$$\left. \frac{\partial \log \mathcal{Z}_{N_0}}{\partial \epsilon_0} \right|_{\epsilon_{\max}} = -\sigma \{ \log[1 + \exp(\beta(\epsilon_{\max} - \mu))] - (\sigma k_B T)^{-1} \} = 0, \quad (3)$$

and for $\sigma k_B T \gg 1$ [16]

$$\epsilon_{\max} = \mu - k_B T \log(\sigma k_B T). \quad (4)$$

We observe here that $\beta(\mu - \epsilon_{\max})$ depends only on $\sigma k_B T$ as long as $\mu > k_B T \log(\sigma k_B T)$. Therefore, as T decreases and μ becomes larger than $k_B T \log(\sigma k_B T)$, the probability distribution (1) forms a maximum at $N_0 > 0$. We say that at this temperature the condensate starts to form and equation $\mu = k_B T \log(\sigma k_B T)$ defines the condensation temperature [15, 16].

At low temperatures, the maximum of w_{N_0} becomes sharp and ϵ_0 approaches μ . In this temperature range we shall approximate w_{N_0} around the maximum by a Gaussian distribution,

$$w_{N_0} \approx w(N_{\max}) \exp \left[-\frac{(N_0 - N_{\max})^2}{2\Delta^2} \right] \quad (5)$$

of width

$$\Delta^{-2} = -\left. \frac{\partial^2 \log \mathcal{Z}_{N_0}}{\partial N_0^2} \right|_{N_{\max}} = (\sigma k_B T)^{-1} - (1 + \sigma k_B T)^{-1} \approx (\sigma k_B T)^{-2}. \quad (6)$$

In this case, one can approximate \mathcal{Z}_{N_0} by $\mathcal{Z}_{N_0}^{(a)}$ given by the expression

$$\mathcal{Z}_{N_0}^{(a)} = \exp \left\{ \log \mathcal{Z}_{N_{\max}} + \frac{1}{2} \frac{\partial^2 \log \mathcal{Z}_{N_0}}{\partial N_0^2} (\delta N_0)^2 \right\} = \mathcal{Z}_{N_{\max}} \exp \left\{ -\frac{1}{2} \frac{(\delta N_0)^2}{(\sigma k_B T)^2} \right\}. \quad (7)$$

Here, $\delta N_0 \equiv N_0 - N_{\max}$. To check the accuracy of the approximation, we calculate first the total partition function as

$$Z^{(a)} = \int_{-\infty}^{\infty} d(\delta N_0) \mathcal{Z}_{N_0}^{(a)} \approx Z_{N_{\max}} \sqrt{2\pi} \sigma k_B T, \quad (8)$$

where

$$\begin{aligned} \log \mathcal{Z}_{N_{\max}} &= -\beta \left(\sigma \frac{\epsilon_{\max}^2}{2} - \epsilon_{\max} \right) + \beta \mu (\sigma \epsilon_{\max} - 1) + \sigma \int_{\epsilon_{\max}}^{\infty} d\epsilon \log[1 + \exp(-\beta(\epsilon - \mu))] \\ &= \frac{\sigma k_B T}{2} [(\beta \mu)^2 - \log^2(\sigma k_B T)] - \log(\sigma k_B T) + \sigma k_B T \text{Li}_2(-\exp(\beta(\mu - \epsilon_{\max}))), \end{aligned} \quad (9)$$

and Li_2 is Euler's dilogarithm function [7]. Using equation (4) and the expansion

$$\text{Li}_2(-z)|_{z \gg 1} \approx \frac{\log^2|z|}{2} \left(1 + \frac{\pi^2}{6} \frac{2}{\log^2|z|} \right), \quad (10)$$

we obtain

$$\log \mathcal{Z}^{(a)} \approx \frac{\sigma k_B T}{2} (\beta \mu)^2 + \frac{\pi^2}{6} \sigma k_B T + \log \sqrt{2\pi}. \quad (11)$$

The exact partition function,

$$\log \mathcal{Z} = \sigma \int_0^{\infty} d\epsilon \log[1 + \exp(-\beta(\epsilon - \mu))] = \sigma k_B T \text{Li}_2[-\exp(\beta \mu)], \quad (12)$$

using the approximation (10) can be put into the form

$$\log \mathcal{Z} = \frac{\sigma k_B T}{2} (\beta \mu)^2 + \frac{\pi^2}{6} \sigma k_B T. \quad (13)$$

The expansions of $\log \mathcal{Z}$ and $\log \mathcal{Z}^{(a)}$ are identical up to order $\sigma k_B T / \log^2(\sigma k_B T)$. The term $\log \sqrt{2\pi} \approx 0.92$ from equation (11) may be neglected at $\sigma k_B T \gg 1$, since this is smaller than $\sigma k_B T / \log^2(\sigma k_B T) \gg 1$.

From the approximation (5) we obtain

$$\sqrt{\langle \delta^2 N_0 \rangle} = \sigma k_B T. \quad (14)$$

The corresponding canonical fluctuation, calculated by the saddle-point method applied to the equivalent Bose system [17, 18], is

$$\sqrt{\langle \delta^2 N_0 \rangle^c} = \sqrt{\zeta(2)} \sigma k_B T. \quad (15)$$

Obviously, the two analytical approximations, (14) and (15), do not coincide and the question that remains to be answered is whether these distributions are indeed different, or simply the Gaussian approximation (5) is not good enough.

3. Numerical evaluation of the fluctuations

In this section, we calculate the fluctuations numerically by introducing a recursion relation. From equation (1), we obtain

$$\frac{w_{N_0+1}}{w_{N_0}} = \frac{\exp[-\beta(\epsilon_{N_0} - \mu)]}{1 + \exp[-\beta(\epsilon_{N_0+1} - \mu)]} \quad (16)$$

and the value of N_{\max} may be found by solving

$$\frac{\exp[-\beta(\epsilon_{N_{\max}} - \mu)]}{1 + \exp[-\beta(\epsilon_{N_{\max}+1} - \mu)]} = 1. \quad (17)$$

If the density of states is constant and $\epsilon_{i+1} - \epsilon_i = \sigma^{-1}$ for any i , then equation (17) becomes

$$\frac{\exp[-\beta(\epsilon_{\max} - \mu)]}{1 + \exp[-\beta(\epsilon_{\max} + \sigma^{-1} - \mu)]} = 1. \quad (18)$$

Using equations (16) and (18), we may now calculate numerically N_{\max} , $\langle N_0 \rangle$ and $\langle \delta^2 N_0 \rangle$. If $\sigma k_B T \gg 1$, by writing $\exp[-\beta(\epsilon_{\max} + \sigma^{-1} - \mu)] \approx \exp[-\beta(\epsilon_{\max} - \mu)](1 - (\sigma k_B T)^{-1})$ equation (18) may be simplified to $\exp[\beta(\mu - \epsilon_{\max})] = \sigma k_B T$, which is the same as equation (4). Moreover, since around the maximum $\exp[\beta(\mu - \epsilon_{\max})] \gg 1$, in the relevant energy interval we may transform equation (16) into

$$\frac{w_{N_0+1}}{w_{N_0}} = [\exp(\beta(\epsilon_{N_0} - \mu)) + \exp(-1/\sigma k_B T)]^{-1} \approx 1 - \exp(\beta(\epsilon_{N_0} - \mu)) + \frac{1}{\sigma k_B T}. \quad (19)$$

Let us now analyse the equivalent Bose gas with constant density of states. The canonical partition function for a system of N_{ex} particles is [17]

$$Z_{N_{\text{ex}}}^{\text{b}} = \prod_{k=1}^{N_{\text{ex}}} (1 - q^k)^{-1}, \quad q \equiv \exp(-1/\sigma k_B T). \quad (20)$$

In a canonical system of N particles, the probability $w_{N_{\text{ex}}}^{\text{b}}$ to have exactly N_{ex} particles in the excited states is proportional to $Z_{N_{\text{ex}}}^{\text{b}} - Z_{N_{\text{ex}}-1}^{\text{b}}$ [18], so we have

$$w_{N_{\text{ex}}}^{\text{b}} = \frac{q^{N_{\text{ex}}}}{Z} \prod_{k=1}^{N_{\text{ex}}} (1 - q^k)^{-1}. \quad (21)$$

Since $N_{\text{ex}} \equiv N - N_0$, the relative probability which corresponds to equation (16) for fermions is

$$\frac{w_{N_{\text{ex}}-1}^{\text{b}}}{w_{N_{\text{ex}}}^{\text{b}}} = \frac{1 - q^{N_{\text{ex}}}}{q} = (1 - \exp(N_{\text{ex}}/\sigma k_B T)) e^{1/\sigma k_B T}. \quad (22)$$

The most probable N_{ex} is then given by equating right-hand side of equation (22) to 1.

We want now to compare equations (16) and (22) in the limit $\sigma k_B T \gg 1$. For this we take a Fermi and a Bose system with the same number of particles, N . In the Fermi system we define the Fermi energy, $\epsilon_F = N/\sigma$. We shall assume that both systems are below the condensation temperature and the number of particles in the condensate is N_0 . Above the condensate we have N_{ex} particles. For a condensed gas $\epsilon_F - \mu < \sigma^{-1}$, so we can express N_{ex} in equation (22) as $N_{\text{ex}} = \sigma(\epsilon_F - \epsilon_0) = \sigma(\mu - \epsilon_0)$. By doing so, equation (22) becomes

$$\frac{w_{N_0+1}^{\text{b}}}{w_{N_0}^{\text{b}}} = (1 - \exp(\beta(\epsilon_{n_0} - \mu))) \exp(1/\sigma k_B T) \approx 1 - \exp(\beta(\epsilon_{n_0} - \mu)) + \frac{1}{\sigma k_B T}, \quad (23)$$

which is identical to equation (19). Therefore, the two probability distributions w_{N_0} and $w_{N_0}^{\text{b}}$ approach each other in the limit of large systems, i.e. when $\sigma k_B T \gg 1$.

The numerical calculations, based on equations (16) and (22) are plotted in figure 2. We already observe that for $\sigma k_B T \geq 1$, the fluctuation of the particle number in the condensate is almost the same as for the canonical Bose and the grandcanonical Fermi systems. This justifies the approach taken in [16], and for $\sigma k_B T \gg 1$, N_0 may be calculated directly as the average number of particles in the Bose condensate, rather than by equation (3).

Notable *relative* differences between the canonical and the grandcanonical results appear only for $\sigma k_B T$ about 1 or below. For these values of $\sigma k_B T$ the fluctuations depend on the exact location of μ , with respect to the single-particle levels. For example, let us say

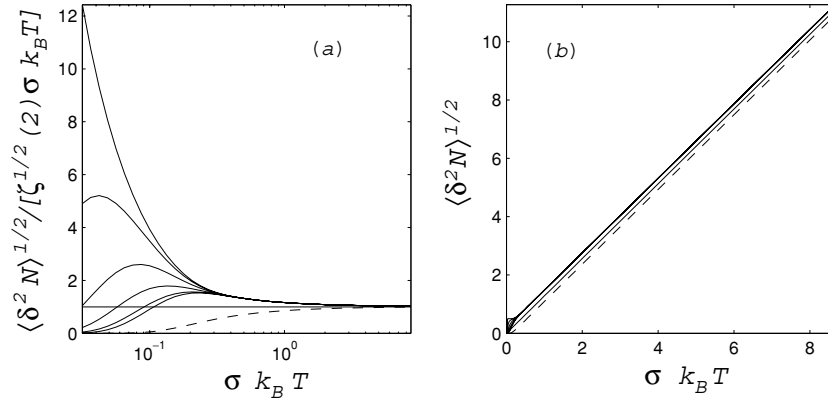


Figure 2. Numerical calculations of $\langle \delta^2 N \rangle^{1/2} / [\zeta^{1/2}(2)\sigma k_B T]$ (a) and $\langle \delta^2 N \rangle^{1/2}$ (b). For low values of $\sigma k_B T$, $\langle \delta^2 N \rangle$ depends on the exact location of μ , with respect to the single-particle levels. If ϵ_N and ϵ_{N+1} are two consecutive energy levels ($\epsilon_{N+1} - \epsilon_N = \sigma^{-1}$), then the solid curves in both (a) and (b) figures correspond to $\mu = \epsilon_N + 0.1 \cdot i \sigma^{-1}$ ($i = 0, 1, \dots, 5$) from top to bottom. The dashed lines represent the canonical fluctuations and the straight lines ($y = 1$ in (a) and $y = \zeta^{1/2}(2)x$ in (b)) correspond to the asymptotic values.

that $\mu \in (\epsilon_{N-1}, \epsilon_N)$, where ϵ_{N-1} and ϵ_N are two consecutive energy levels. In the limit $\beta(\mu - \epsilon_{N-1}) \rightarrow \infty$ and for $N_0 = N$, equation (16) becomes

$$\frac{w_N}{w_{N-1}} \approx \exp(\beta(\mu - \epsilon_{N-1})). \quad (24)$$

For $\beta(\mu - \epsilon_{N-1}) \rightarrow \infty$, we can calculate $\langle N_0 \rangle$ and $\langle \delta^2 N_0 \rangle$ by taking into account only the levels ϵ_{N-1} and ϵ_N . We obtain

$$\langle N_0 \rangle = \frac{\exp[\beta(\mu - \epsilon_{N-1})](N+1) + N}{\exp[\beta(\mu - \epsilon_{N-1})] + 1} = N+1 - \frac{1}{\exp[\beta(\mu - \epsilon_{N-1})] + 1}, \quad (25)$$

$$\begin{aligned} \langle \delta^2 N_0 \rangle &= \frac{\exp[\beta(\mu - \epsilon_{N-1})] \exp[-2\beta(\mu - \epsilon_{N-1})] + (1 - \exp[-\beta(\mu - \epsilon_{N-1})])^2}{\exp[\beta(\mu - \epsilon_{N-1})] + 1} \\ &= \frac{1 - \exp[-\beta(\mu - \epsilon_{N-1})]}{\exp[\beta(\mu - \epsilon_{N-1})] + 1} \approx \exp[-\beta(\mu - \epsilon_{N-1})]. \end{aligned} \quad (26)$$

Therefore, for any $\mu \in (\epsilon_N, \epsilon_{N+1})$,

$$\lim_{T \rightarrow 0} \{ \langle \delta^2 N_0 \rangle^{1/2} / \zeta^{1/2}(2)\sigma k_B T \} = 0. \quad (27)$$

The situation is different if, for example, $\mu = \epsilon_N$. Then, applying the same algorithm as above, we get $\langle N_0 \rangle = N + 0.5$ and $\langle \delta^2 N \rangle^{1/2} = 0.5$. In figure 2, we plotted $\langle \delta^2 N \rangle^{1/2} / [\zeta^{1/2}(2)\sigma k_B T]$ (a) and $\langle \delta^2 N \rangle^{1/2}$ (b) for $\mu = \epsilon_N + 0.1 \cdot i \sigma^{-1}$ ($i = 0, 1, \dots, 5$). The fluctuations normalized to the asymptotic value, $\zeta^{1/2}(2)\sigma k_B T$, are quite different for $\sigma k_B T \leq 1$, but the absolute values of the fluctuations are very close for any $\sigma k_B T$ for both types of systems and any choice of μ .

From figure 2(b), we also note that although the difference $\sqrt{\langle \delta^2 N_0 \rangle} - \sqrt{\langle \delta^2 N_0 \rangle^c}$ is very small for any $\sigma k_B T$, it does not tend to zero as $\sigma k_B T \rightarrow \infty$. Numerically, we obtain

$$\sqrt{\langle \delta^2 N_0 \rangle} - \sqrt{\langle \delta^2 N_0 \rangle^c} \approx 0.39 \quad \text{for } \sigma k_B T \gg 1. \quad (28)$$

To explain this difference, let us note that the fluctuation of N_0 in the grandcanonical ensemble, $\delta N_0 \equiv N_0 - \langle N_0 \rangle$, may be viewed as the superposition of fluctuations coming from two

sources: the canonical fluctuation of N_0 around its average value, corresponding to the total particle number N , denoted by $\delta N_0^c (\equiv N_0 - \langle N_0 \rangle_N)$, and the fluctuation of $\langle N_0 \rangle_N$ due to the grandcanonical fluctuation of N . Assuming small fluctuations, the variation of $\langle N_0 \rangle_N$ due to the variation of N may be written as

$$\delta \langle N_0 \rangle_N = \frac{\partial \langle N_0 \rangle_N}{\partial N} \delta N.$$

Collecting all these together, we write

$$\delta N_0 = N_0 - \langle N_0 \rangle \equiv N_0 - \langle N_0 \rangle_N + \langle N_0 \rangle_N - \langle N_0 \rangle = \delta N_0^c + \frac{\partial \langle N_0 \rangle}{\partial N} \delta N. \quad (29)$$

Below the condensation, $\partial \langle N_0 \rangle / \partial N = 1$ (temperature stays constant). Moreover, well below the condensation temperature, like in the Maxwell Demon's ensemble [19], δN_0^c and δN are independent, since the condensate may be viewed as a reservoir of particles. With these clarifications, equation (29) leads to

$$\langle \delta^2 N_0 \rangle = \langle \delta^2 N_0 \rangle^c + \langle \delta^2 N \rangle. \quad (30)$$

For high enough $\sigma k_B T$ and $\beta \mu$, we take $\langle \delta^2 N \rangle = \sigma k_B T$, which, if plugged into equation (30) yields

$$\langle \delta^2 N_0 \rangle \approx \sqrt{\zeta(2)(\sigma k_B T)^2 + \sigma k_B T} \approx \sqrt{\zeta(2)} \sigma k_B T + [2\sqrt{\zeta(2)}]^{-1}. \quad (31)$$

As we expect, $([2\sqrt{\zeta(2)}]^{-1}) \approx 0.39$.

Using the same method, we calculate the fluctuation of the number of particles in the thermally active layer without doing any extra numerics. Again we denote by $\langle N_{\text{ex}} \rangle$ the average number of particles in the thermally active layer and the fluctuation δN_{ex} can again be written as

$$\delta N_{\text{ex}} \equiv N_{\text{ex}} - \langle N_{\text{ex}} \rangle = N_{\text{ex}} - \langle N_{\text{ex}} \rangle_N + \langle N_{\text{ex}} \rangle_N - \langle N_{\text{ex}} \rangle = \delta^c N_{\text{ex}} + \frac{\partial \langle N_{\text{ex}} \rangle}{\partial N} \delta N. \quad (32)$$

By $\langle N_{\text{ex}} \rangle_N$ we denote the average number of particles in the thermally active layer at fixed N . Well below the condensation temperature $\langle N_{\text{ex}} \rangle_N$ does not depend on N , so from equation (32), we get $\delta N_{\text{ex}} = \delta^c N_{\text{ex}}$ and

$$\langle \delta^2 N_{\text{ex}} \rangle = \langle \delta^2 N_{\text{ex}} \rangle^c = \langle \delta^2 N_0 \rangle^c. \quad (33)$$

4. Conclusions

The thermodynamic equivalence between ideal bosons and fermions with the same constant density of states σ [1–5, 8, 10, 13] is apparently lost below the Bose–Einstein condensation temperature [9], T_c . On the other hand, it was proven that if the Bose and the Fermi systems have the same spectrum consisting of nondegenerate, equidistant single-particle states (like, for example, particles in a one-dimensional harmonic potential), then the canonical thermodynamic equivalence between the two systems is preserved down to zero temperature in the smallest details [10, 13]. This apparent contradiction is due to the fact that below T_c , the condensate is formed in the Bose gas and to this it corresponds a degenerate subsystem in the Fermi gas that was previously overlooked. The existence of the degenerate subsystem changes slightly the values of intensive parameters, like the chemical potential, in such a way that the equivalence is restored. Because of the correspondence between the Fermi degenerate subsystem and the Bose–Einstein condensate, we called the first one Fermi condensate. The particles above the Fermi condensate form the so-called thermally active layer.

One can emphasize the Fermi condensate by calculating in the grandcanonical ensemble the probability to have N_0 degenerate particles at the bottom of the single-particle spectrum

(see equation (1)). Below T_c this probability distribution has a maximum for $N_0 > 0$ and we showed numerically and analytically in section 3 that the grandcanonical average of N_0 is the same as the canonical average.

We also calculated the fluctuation of the condensate and of the particles in the thermally active layer. The grandcanonical fluctuation of N_0 is almost the same as the canonical fluctuation. Although the average values $\langle N_0 \rangle$ and $\langle N_0 \rangle^c$ are identical, for large values of $\sigma k_B T$ the fluctuations $\langle \delta^2 N_0 \rangle^{1/2}$ and $(\langle \delta^2 N_0 \rangle^c)^{1/2}$ differ by a small, but a constant value, which is 0.39. This is due to the extra contribution to $\langle \delta^2 N_0 \rangle^{1/2}$, given by the grandcanonical fluctuation of the total particle number.

The fermions in the thermally active layer correspond to the bosons on the excited energy levels. Canonical and grandcanonical averages of N_{ex} are the same. Moreover, well below the condensation temperature, where Maxwell Demon's ensemble is applicable [19], the fluctuation of N_{ex} is the same in both the canonical and the grandcanonical ensembles (see (33)).

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